Dynamic response of an RC beam to a drop weight impact (steel ball)

Ball mass - $M_s = 2.1\,\mathrm{t}$

Ball material - steel

Modulus of elasticity - $E_s = 206\,\mathrm{GPa}$

Poisson's ratio - $\nu_s = 0.3$

Mass density - $\rho_s = 7.85\,\dfrac{\mathrm{t}}{\mathrm{m}^3}$

Ball volume - $V_s = \dfrac{M_s}{\rho_s} = \dfrac{2.1\,\mathrm{t}}{7.85\,\mathrm{t/m}^3} = 0.268\,\mathrm{m}^3$ = 4/3πR³

Ball radius - $R_s = \sqrt[3]{\dfrac{3}{4}\cdot\dfrac{V_s}{\pi}} = 400\,\mathrm{mm}$

Height of bottom above the beam surface - $H = 2\,\mathrm{m}$

Structure type - simply supported beam

Beam length - $L = 12\,\mathrm{m}$

Material - reinforced concrete C20/25

Modulus of elasticity - $E = 20\,\mathrm{GPa}$

Poisson's ratio - $\nu = 0.2$

Shear modulus - $G = \dfrac{E}{2\cdot(1+\nu)} = \dfrac{20\,\mathrm{GPa}}{2\cdot(1+0.2)} = 8.33\,\mathrm{GPa}$

Unit weight - $\gamma_b = 25\,\dfrac{\mathrm{kN}}{\mathrm{m}^3}$

Cross section - rectangular with dimensions:

Width - $b = 350\,\mathrm{mm}$

Height - $h = 650\,\mathrm{mm}$

Area - $A = b\cdot h = 350\,\mathrm{mm}\cdot 650\,\mathrm{mm} = 2275\,\mathrm{cm}^2$

Second moment of area - $I = \dfrac{b\cdot h^3}{12} = \dfrac{350\,\mathrm{mm}\cdot(650\,\mathrm{mm})^3}{12} = 8009895833\,\mathrm{mm}^4$

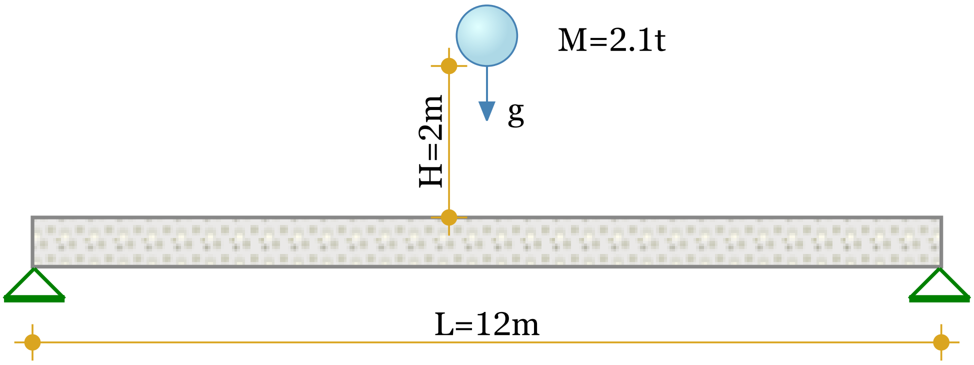
Shear area - $A_Q = \dfrac{5}{6}\cdot A = \dfrac{5}{6}\cdot 2275\,\mathrm{cm}^2 = 1895.83\,\mathrm{cm}^2$

Self-weight - $g_b = A\cdot\gamma_b = 2275\,\mathrm{cm}^2\cdot 25\,\mathrm{kN/m}^3 = 5.69\,\mathrm{kN/m}$

Uniform load - $q = 10\,\dfrac{\mathrm{kN}}{\mathrm{m}}$

Gravity acceleration - $g = 9.81\,\dfrac{\mathrm{m}}{\mathrm{s}^2}$

Uniform mass - $m = \dfrac{g_b + q}{g} = \dfrac{5.69\,\mathrm{kN/m} + 10\,\mathrm{kN/m}}{9.81\,\mathrm{m/s}^2} = 1.6\,\mathrm{t/m}$



Simple analytical solution

The structure is reduced to a SDOF system for simplicity

Dynamically equivalent mass - $M_d = \frac{2 \cdot L}{\pi} \cdot m = \frac{2 \cdot 12\ \mathrm{m}}{3.14} \cdot 1.6\ \mathrm{t/m} = 12.22\ \mathrm{t}$

Potential energy of the ball before dropping

$E_p = M_s \cdot g \cdot H = 2.1\ \mathrm{t} \cdot 9.81\ \mathrm{m/s}^2 \cdot 2\ \mathrm{m} = 41.28\ \mathrm{kJ}$

Kinetic energy immediately before the impact - $E_k = \frac{M_s \cdot v_0^{\,2}}{2}$

The velocity at the moment before the impact is determined by the energy conservation law $E_k = E_p$:

$$v_0 = \sqrt{\frac{2 \cdot E_p}{M_s}} = \sqrt{\frac{2 \cdot 41.28\ \mathrm{kJ}}{2.1\ \mathrm{t}}} = 6.26\ \mathrm{m/s}$$

Perfectly inelastic collision model is assumed.

Total mass after contact - $M_{tot} = M_s + M_d = 2.1\ \mathrm{t} + 12.22\ \mathrm{t} = 14.33\ \mathrm{t}$

The velocity immediately after the contact is determined by the law of conservation of momentum:

$$v_1 = \frac{v_0 \cdot M_s}{M_{tot}} = \frac{6.26\ \mathrm{m/s} \cdot 2.1\ \mathrm{t}}{14.33\ \mathrm{t}} = 0.92\ \mathrm{m/s}$$

Structural stiffness for a vertical force applied at the middle point of the span

$$K = \frac{48 \cdot E \cdot I}{L^3} = \frac{48 \cdot 20\ \mathrm{GPa} \cdot 8009895833\ \mathrm{mm}^4}{(12\ \mathrm{m})^3} = 4449.94\ \mathrm{kN/m}$$

Deflection due to uniform load

$$z_0 = \frac{5 \cdot (g_b + q) \cdot L^4}{384 \cdot E \cdot I} = \frac{5 \cdot (5.69\ \mathrm{kN/m} + 10\ \mathrm{kN/m}) \cdot (12\ \mathrm{m})^4}{384 \cdot 20\ \mathrm{GPa} \cdot 8009895833\ \mathrm{mm}^4} = 26.44\ \mathrm{mm}$$

Static displacement - $z_{st} = \frac{M_{tot} \cdot g}{K} = \frac{14.33\ \mathrm{t} \cdot 9.81\ \mathrm{m/s}^2}{4449.94\ \mathrm{kN/m}} = 31.57\ \mathrm{mm}$

Natural circular frequency - $\omega_1 = \sqrt{\frac{K}{M_{tot}}} = \sqrt{\frac{4449.94\ \mathrm{kN/m}}{14.33\ \mathrm{t}}} = 17.62\ \mathrm{s}^{-1}$

Vibration period - $T_1 = \frac{2 \cdot \pi}{\omega_1} = \frac{2 \cdot 3.14}{17.62\ \mathrm{s}^{-1}} = 0.356\ \mathrm{s}$

Dynamic factor

$$\mu = 1 + \sqrt{1 + \left(\frac{v_1 \cdot \omega_1}{g}\right)^2} = 1 + \sqrt{1 + \left(\frac{0.92\ \mathrm{m/s} \cdot 17.62\ \mathrm{s}^{-1}}{9.81\ \mathrm{m/s}^2}\right)^2} = 2.93$$

Dynamic displacement - $z_d = \mu \cdot z_{st} = 2.93 \cdot 31.57\ \mathrm{mm} = 92.58\ \mathrm{mm}$

Dynamic force - $F_d = \mu \cdot M_s \cdot g = 2.93 \cdot 2.1\ \mathrm{t} \cdot 9.81\ \mathrm{m/s}^2 = 60.52\ \mathrm{kN}$

(without self-weight and uniform load)

Simplified equation for the dynamic factor

$$\mu_1 = 1 + \frac{v_1 \cdot \omega_1}{g} = 1 + \frac{0.92\ \mathrm{m/s} \cdot 17.62\ \mathrm{s}^{-1}}{9.81\ \mathrm{m/s}^2} = 2.65$$

The difference will be smaller for greater heights.

Elastic time history response of the structure as an SDOF system

Damped vibration is assumed with factor - $\xi = 0.05$

Vibration amplitude - $A = z_d - z_{st} = 92.58\ \mathrm{mm} - 31.57\ \mathrm{mm} = 61.01\ \mathrm{mm}$ or

$$A = \frac{v_1}{\omega_1} = \frac{0.92\ \mathrm{m/s}}{17.62\ \mathrm{s}^{-1}} = 52.21\ \mathrm{mm}$$

Theoretical equation of motion

$$y(t) = A \cdot e^{-\xi \cdot \omega_1 \cdot t} \cdot \sin(\omega_1 \cdot t)$$

Solution by direct integration of the impulse load

Duration of impulse transmission for a beam with infinite mass [1]

$$\tau_L = 2.94 \cdot \sqrt[5]{\frac{\left(\frac{15}{16} \cdot M_s \cdot \left(\frac{1-\nu^2}{E} + \frac{1-\nu_s^{\,2}}{E_s}\right)\right)^2}{R_s \cdot v_0}} = 2.94 \cdot \sqrt[5]{\frac{\left(\frac{15}{16} \cdot 2.1\ \mathrm{t} \cdot \left(\frac{1-0.2^2}{20\ \mathrm{GPa}} + \frac{1-0.3^2}{206\ \mathrm{GPa}}\right)\right)^2}{400\ \mathrm{mm} \cdot 6.26\ \mathrm{m/s}}} = 3.93\ \mathrm{ms}$$

Duration of impulse transmission for a beam with finite mass [2]

$$\tau_L = 2.94 \cdot \sqrt[5]{\frac{\left(\frac{15}{16} \cdot \frac{M_s \cdot M_d}{M_s + M_d} \cdot \left(\frac{1-\nu^2}{E} + \frac{1-\nu_s^{\,2}}{E_s}\right)\right)^2}{R_s \cdot v_0}} = 2.94 \cdot \sqrt[5]{\frac{\left(\frac{15}{16} \cdot \frac{2.1\ \mathrm{t} \cdot 12.22\ \mathrm{t}}{2.1\ \mathrm{t} + 12.22\ \mathrm{t}} \cdot \left(\frac{1-0.2^2}{20\ \mathrm{GPa}} + \frac{1-0.3^2}{206\ \mathrm{GPa}}\right)\right)^2}{400\ \mathrm{mm} \cdot 6.26\ \mathrm{m/s}}} = 3.69\ \mathrm{ms}$$

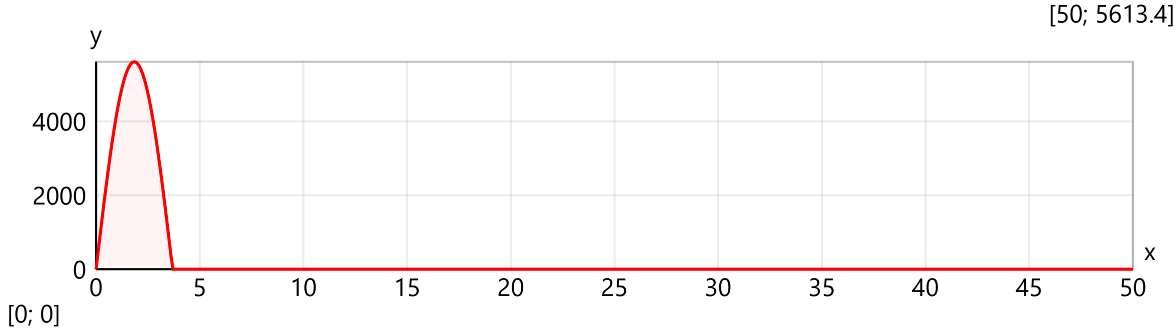
The above values correspond well to the experimental data in [3], where the recorded durations are of a similar magnitude.

The impulse force function will be determined by using the recommended expressions (9.20) - (9.22) in [1]

The coefficient of restitution for perfectly inelastic collision is - $e. = 0$

$$F(t) = M_s \cdot v_0 \cdot (1 + e.) \cdot \frac{\pi}{2 \cdot \tau_L} \cdot \sin\left(\frac{\pi}{\tau_L} \cdot t\right) \cdot (|t| \le \tau_L)$$

Impulse load diagram



Maximal impulse load value - $F_{max} = F\left(\frac{\tau_L}{2}\right) = F\left(\frac{3.69\ \mathrm{ms}}{2}\right) = 5613.66\ \mathrm{kN}$

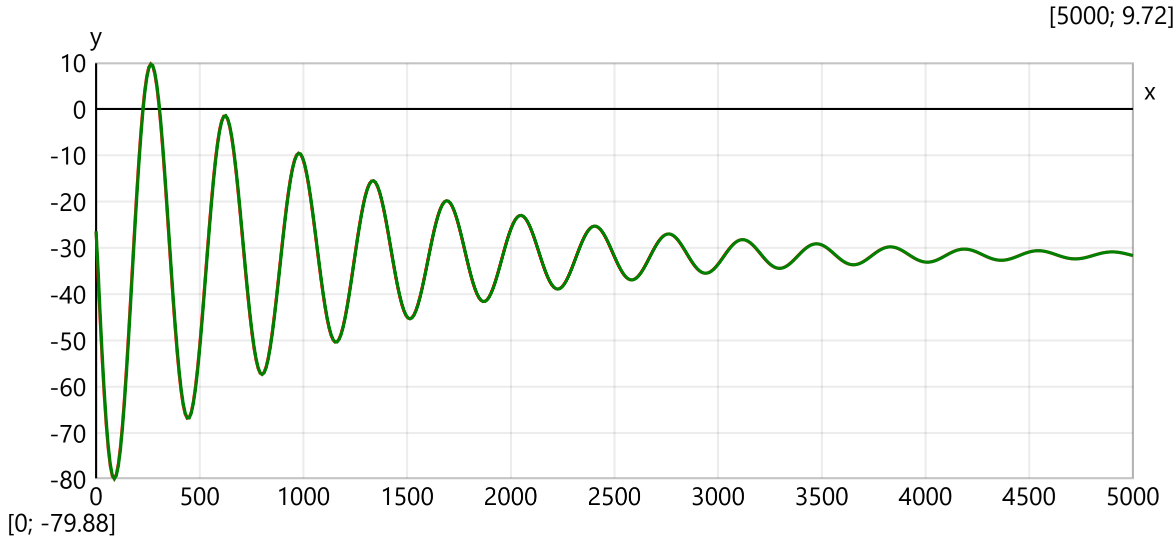
The equation of motion is expressed by the Duhamel's integral

$$y_D(t) = \frac{1}{M_{tot} \cdot \omega_1} \cdot \int_{0\,\mathrm{ms}}^{\min(t;\ \tau_L)} F(\tau) \cdot e^{-\xi \cdot \omega_1 \cdot (t - \tau)} \cdot \sin(\omega_1 \cdot (t - \tau))\, d\tau$$

Static displacement for the midpoint of the beam

$$y_0(t) = z_0 + (z_{st} - z_0) \cdot \begin{cases} \text{if } t < \frac{T_1}{4} : \sin\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) \\ \text{else: } 1 \end{cases}$$

Time history of the midpoint displacement, [mm]



Elastic time history response of the structure as an MDOF system

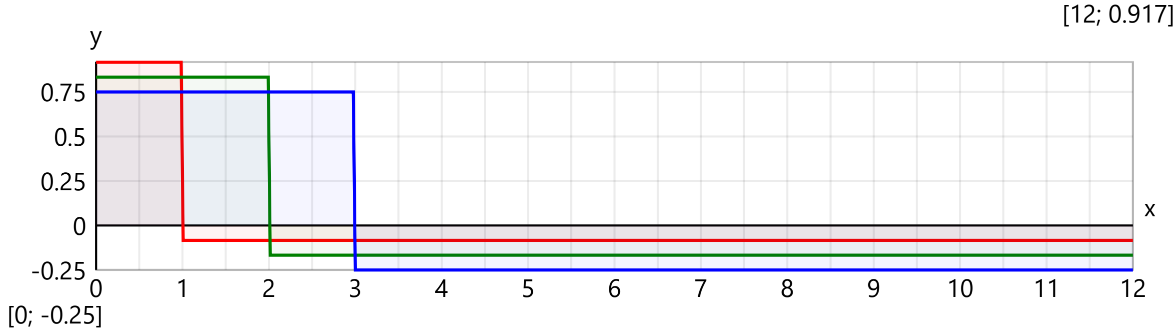
Number of intermediate joints - $n_J = 11$ (odd)

Length of one segment - $\Delta x = \frac{L}{n_J + 1} = \frac{12\ \mathrm{m}}{11 + 1} = 1\ \mathrm{m}$

Coordinate of joint *j* - $x_J(j) = \Delta x \cdot j$

Shear forces due to unit vertical load $F_j = 1$ at joint *j*

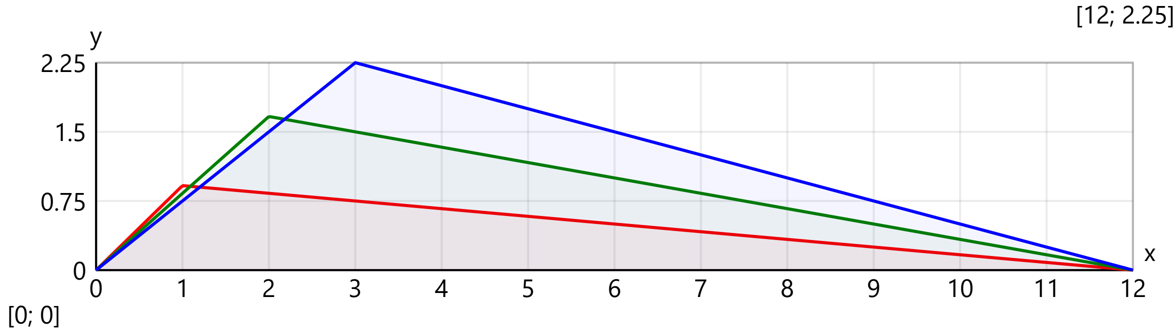
$$V_1(x;j) = \begin{cases} \text{if } x < x_J(j): 1 - \dfrac{x_J(j)}{L} \\ \text{else: } \dfrac{-x_J(j)}{L} \end{cases}$$



Bending moments due to unit vertical load $F_j = 1$ at joint j

$$M_{1,\max}(j) = \left(\frac{x_J(j)}{L} - 1\right)\cdot x_J(j)$$

$$M_1(x;j) = M_{1,\max}(j)\cdot\begin{cases} \text{if } x < x_J(j): \dfrac{x}{x_J(j)} \\ \text{else: } \dfrac{L - x}{L - x_J(j)} \end{cases}$$



Flexibility matrix

$$D(i;j) = \left(\int_{0\,\mathrm{m}}^{L} M_1(x;i)\cdot M_1(x;j)\,\mathrm{d}x\right)\cdot\frac{1}{E\cdot I} + \left(\int_{0\,\mathrm{m}}^{L} V_1(x;i)\cdot V_1(x;j)\,\mathrm{d}x\right)\cdot\frac{1}{G\cdot A_Q}$$

$$D = \begin{bmatrix}
0.0216 & 0.0378 & 0.0489 & 0.0552 & 0.0574 & 0.056 & 0.0514 & 0.0443 & 0.035 & 0.0242 & \cdots & 0.0124 \\
0.0378 & 0.0704 & 0.093 & 0.106 & 0.111 & 0.109 & 0.1 & 0.0864 & 0.0685 & 0.0474 & \cdots & 0.0242 \\
0.0489 & 0.093 & 0.128 & 0.149 & 0.158 & 0.155 & 0.144 & 0.124 & 0.0988 & 0.0685 & \cdots & 0.035 \\
0.0552 & 0.106 & 0.149 & 0.179 & 0.193 & 0.193 & 0.18 & 0.156 & 0.124 & 0.0864 & \cdots & 0.0443 \\
0.0574 & 0.111 & 0.158 & 0.193 & 0.214 & 0.217 & 0.205 & 0.18 & 0.144 & 0.1 & \cdots & 0.0514 \\
0.056 & 0.109 & 0.155 & 0.193 & 0.217 & 0.227 & 0.217 & 0.193 & 0.155 & 0.109 & \cdots & 0.056 \\
0.0514 & 0.1 & 0.144 & 0.18 & 0.205 & 0.217 & 0.214 & 0.193 & 0.158 & 0.111 & \cdots & 0.0574 \\
0.0443 & 0.0864 & 0.124 & 0.156 & 0.18 & 0.193 & 0.193 & 0.179 & 0.149 & 0.106 & \cdots & 0.0552 \\
0.035 & 0.0685 & 0.0988 & 0.124 & 0.144 & 0.155 & 0.158 & 0.149 & 0.128 & 0.093 & \cdots & 0.0489 \\
0.0242 & 0.0474 & 0.0685 & 0.0864 & 0.1 & 0.109 & 0.111 & 0.106 & 0.093 & 0.0704 & \cdots & 0.0378 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0.0124 & 0.0242 & 0.035 & 0.0443 & 0.0514 & 0.056 & 0.0574 & 0.0552 & 0.0489 & 0.0378 & \cdots & 0.0216
\end{bmatrix}$$

mm/kN

Mass matrix

$$d_{M.j} = \frac{m \cdot \Delta x}{\mathrm{t}} = \frac{1.6\ \mathrm{t/m} \cdot 1\ \mathrm{m}}{\mathrm{t}} = 1.6$$

$$M = \begin{bmatrix} 1.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1.6 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1.6 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.7 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.6 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1.6 \end{bmatrix}$$

Total mass of the structure - $\mathbf{sum}(\vec{d}_M) = 19.7\ \mathrm{t}$

Eigenvalues

$$M_{sq} = \sqrt{M} = \begin{bmatrix} 1.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1.26 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1.26 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.92 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.26 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.26 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.26 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.26 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1.26 \end{bmatrix}$$

$$C = \mathbf{copy}(M_{sq} \cdot D \cdot M_{sq};\ \mathbf{symmetric}(n_J);\ 1;\ 1) = \mathbf{copy}(M_{sq} \cdot D \cdot M_{sq};\ \mathbf{symmetric}(11);\ 1;\ 1) = \begin{bmatrix} 0.0345 & 0.0605 & 0.0781 & 0.0883 & 0.0918 & 0.136 & 0.0822 & 0.0708 & 0.056 & 0.0387 & \cdots & 0.0198 \\ 0.0605 & 0.113 & 0.149 & 0.17 & 0.178 & 0.265 & 0.16 & 0.138 & 0.11 & 0.0758 & \cdots & 0.0387 \\ 0.0781 & 0.149 & 0.204 & 0.238 & 0.252 & 0.378 & 0.23 & 0.199 & 0.158 & 0.11 & \cdots & 0.056 \\ 0.0883 & 0.17 & 0.238 & 0.287 & 0.309 & 0.469 & 0.287 & 0.25 & 0.199 & 0.138 & \cdots & 0.0708 \\ 0.0918 & 0.178 & 0.252 & 0.309 & 0.343 & 0.529 & 0.328 & 0.287 & 0.23 & 0.16 & \cdots & 0.0822 \\ 0.136 & 0.265 & 0.378 & 0.469 & 0.529 & 0.839 & 0.529 & 0.469 & 0.378 & 0.265 & \cdots & 0.136 \\ 0.0822 & 0.16 & 0.23 & 0.287 & 0.328 & 0.529 & 0.343 & 0.309 & 0.252 & 0.178 & \cdots & 0.0918 \\ 0.0708 & 0.138 & 0.199 & 0.25 & 0.287 & 0.469 & 0.309 & 0.287 & 0.238 & 0.17 & \cdots & 0.0883 \\ 0.056 & 0.11 & 0.158 & 0.199 & 0.23 & 0.378 & 0.252 & 0.238 & 0.204 & 0.149 & \cdots & 0.0781 \\ 0.0387 & 0.0758 & 0.11 & 0.138 & 0.16 & 0.265 & 0.178 & 0.17 & 0.149 & 0.113 & \cdots & 0.0605 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.0198 & 0.0387 & 0.056 & 0.0708 & 0.0822 & 0.136 & 0.0918 & 0.0883 & 0.0781 & 0.0605 & \cdots & 0.0345 \end{bmatrix}$$

$\vec{\lambda}$ = **reverse**(**last**(**eigenvals**($C \cdot 10^{-3}$); 7)) = $\left[0.00261 \;\; 0.000137 \;\; 3.32\times10^{-5} \;\; 9.33\times10^{-6} \;\; 4.78\times10^{-6} \;\; 2.17\times10^{-6} \;\; 1.51\times10^{-6}\right]$

Natural circular frequences - $\vec{\omega} = \sqrt{\dfrac{1}{\vec{\lambda}}} = \left[19.57 \;\; 85.55 \;\; 173.53 \;\; 327.32 \;\; 457.58 \;\; 678.76 \;\; 814.79\right] \; s^{-1}$

Natural vibration frequences - $\vec{f} = \dfrac{\vec{\omega}}{2\cdot\pi}\cdot\mathrm{Hz} = \dfrac{\vec{\omega}}{2\cdot3.14}\cdot\mathrm{Hz} = \left[3.11\,\mathrm{Hz} \;\; 13.61\,\mathrm{Hz} \;\; 27.62\,\mathrm{Hz} \;\; 52.09\,\mathrm{Hz} \;\; 72.83\,\mathrm{Hz} \;\; 108.03\,\mathrm{Hz} \;\; 129.68\,\mathrm{Hz}\right]$

Natural vibration periods - $\vec{T} = \dfrac{1}{\vec{f}} = \left[0.321\,\mathrm{s} \;\; 0.0734\,\mathrm{s} \;\; 0.0362\,\mathrm{s} \;\; 0.0192\,\mathrm{s} \;\; 0.0137\,\mathrm{s} \;\; 0.00926\,\mathrm{s} \;\; 0.00771\,\mathrm{s}\right]$

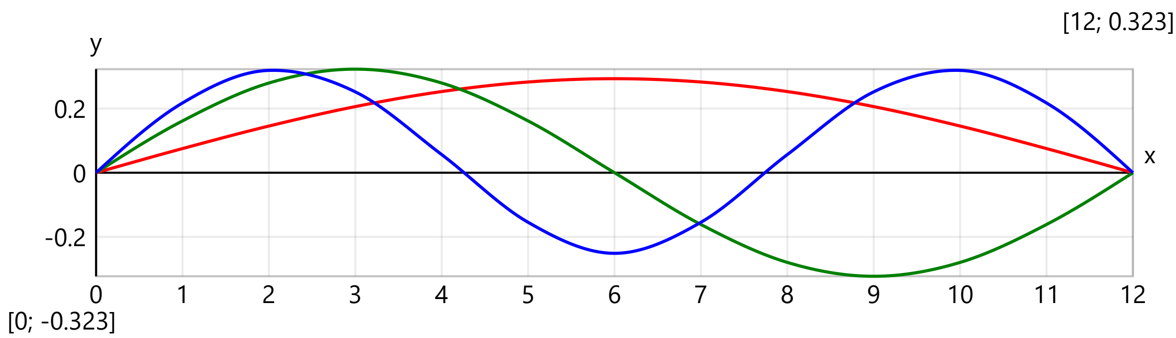
Eigenvectors

Φ = **inverse**(M_{sq})·**extract**$_{cols}$(**eigenvecs**($C\cdot10^{-3}$); **range**(n_J; $n_J - 7 + 1$; -1)) = **inverse**(M_{sq})·**extract**$_{cols}$(**eigenvecs**($C\cdot10^{-3}$); **range**(11; 11 − 7 + 1; -1)) =

$$\begin{bmatrix}
0.0752 & 0.161 & 0.217 & 0.28 & 0.301 & 0.323 & 0.309 \\
0.145 & 0.28 & 0.319 & 0.28 & 0.186 & -9.74\times10^{-13} & -0.113 \\
0.206 & 0.323 & 0.252 & 2.84\times10^{-13} & -0.187 & -0.323 & -0.266 \\
0.253 & 0.28 & 0.0565 & -0.28 & -0.309 & 1.15\times10^{-13} & 0.216 \\
0.283 & 0.161 & -0.155 & -0.28 & -0.027 & 0.323 & 0.209 \\
0.293 & 0 & -0.251 & -7.46\times10^{-15} & 0.22 & 5.01\times10^{-14} & -0.193 \\
0.283 & -0.161 & -0.155 & 0.28 & -0.027 & -0.323 & 0.209 \\
0.253 & -0.28 & 0.0565 & 0.28 & -0.309 & -2.16\times10^{-13} & 0.216 \\
0.206 & -0.323 & 0.252 & -2.77\times10^{-13} & -0.187 & 0.323 & -0.266 \\
0.145 & -0.28 & 0.319 & -0.28 & 0.186 & 1.09\times10^{-12} & -0.113 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0.0752 & -0.161 & 0.217 & -0.28 & 0.301 & -0.323 & 0.309
\end{bmatrix}$$

X = **stack**(**matrix**(1; 3); Φ; **matrix**(1; 3)) =

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0752 & 0.161 & 0.217 & 0.28 & 0.301 & 0.323 & 0.309 \\
0.145 & 0.28 & 0.319 & 0.28 & 0.186 & -9.74\times10^{-13} & -0.113 \\
0.206 & 0.323 & 0.252 & 2.84\times10^{-13} & -0.187 & -0.323 & -0.266 \\
0.253 & 0.28 & 0.0565 & -0.28 & -0.309 & 1.15\times10^{-13} & 0.216 \\
0.283 & 0.161 & -0.155 & -0.28 & -0.027 & 0.323 & 0.209 \\
0.293 & 0 & -0.251 & -7.46\times10^{-15} & 0.22 & 5.01\times10^{-14} & -0.193 \\
0.283 & -0.161 & -0.155 & 0.28 & -0.027 & -0.323 & 0.209 \\
0.253 & -0.28 & 0.0565 & 0.28 & -0.309 & -2.16\times10^{-13} & 0.216 \\
0.206 & -0.323 & 0.252 & -2.77\times10^{-13} & -0.187 & 0.323 & -0.266 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$



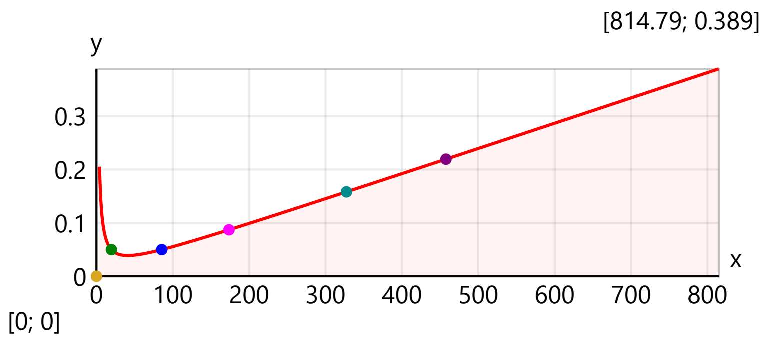
Modal masses - $\vec{m}_\Phi = \mathbf{diag2vec}(\mathbf{transp}(\Phi)\cdot M\cdot\Phi)\cdot \mathrm{t} = [1\,\mathrm{t}\ \ 1\,\mathrm{t}\ \ 1\,\mathrm{t}\ \ 1\,\mathrm{t}\ \ 1\,\mathrm{t}\ \ 1\,\mathrm{t}\ \ 1\,\mathrm{t}]$

Rayleigh damping model is assumed

$\beta = \dfrac{2\cdot\xi}{\vec{\omega}_1 + \vec{\omega}_2} = \dfrac{2\cdot 0.05}{19.57 + 85.55} = 0.000951$, $\alpha = \beta\cdot\vec{\omega}_1\cdot\vec{\omega}_2 = 0.000951\cdot 19.57\cdot 85.55 =$ 1.59

$$\xi(\omega) = \frac{\alpha}{2\cdot\omega} + \frac{\beta\cdot\omega}{2}$$

Modal damping factors - $\vec{\xi}_\Phi = \xi(\vec{\omega}) = [0.05\ \ 0.05\ \ 0.0871\ \ 0.158\ \ 0.219\ \ 0.324\ \ 0.389]$



Damped natural frequences

$\vec{\omega}_D = \vec{\omega}\cdot\sqrt{1 - \vec{\xi}_\Phi\text{^}2}\cdot\mathrm{s}^{-1} = [19.54\,\mathrm{s}^{-1}\ \ 85.44\,\mathrm{s}^{-1}\ \ 172.87\,\mathrm{s}^{-1}\ \ 323.2\,\mathrm{s}^{-1}\ \ 446.43\,\mathrm{s}^{-1}\ \ 642.14\,\mathrm{s}^{-1}\ \ 750.77\,\mathrm{s}^{-1}]$

Dynamic load vector

$F_\Phi(i;\,t) = \Phi_{j_{m},\,i}\cdot F(t)$

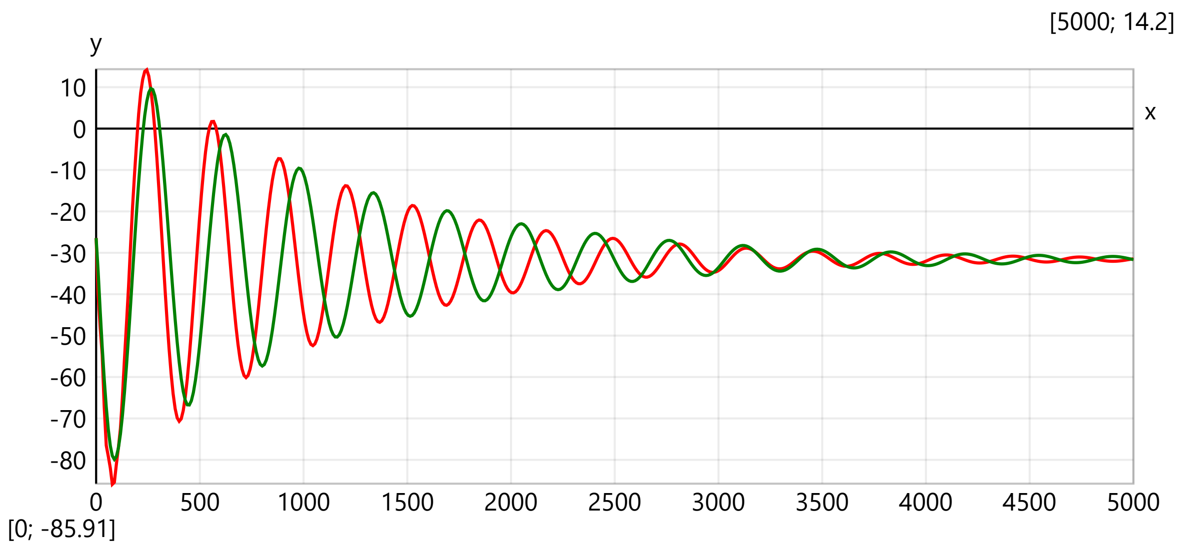
The equations of modal dynamic displacements are expressed by the Duhamel's integral

$$y_\Phi(i;\,t) = \frac{1}{\vec{m}_{\Phi.i}\cdot\omega_{D.i}}\cdot\int_{0\,\mathrm{ms}}^{\min(t;\,\tau_L)} F_\Phi(i;\,\tau)\cdot e^{-\vec{\xi}_{\Phi.i}\cdot\vec{\omega}_i\cdot s^{-1}\cdot(t-\tau)}\cdot\mathbf{sin}(\omega_{D.i}\cdot(t-\tau))\,\mathrm{d}\tau$$

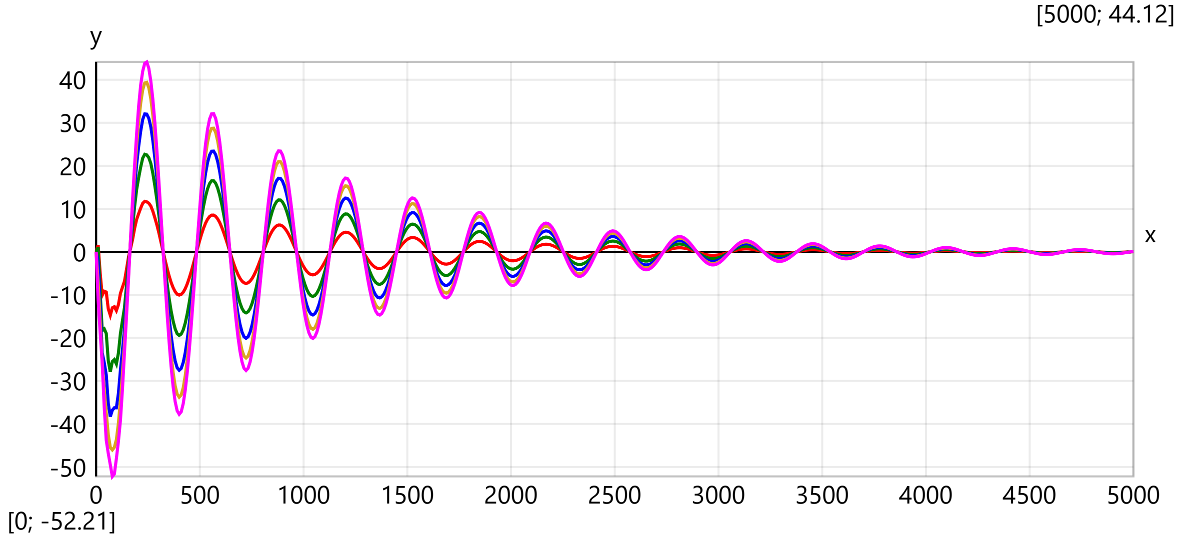
Joint displacements

$$y_J(j;\,t) = \sum_{i=1}^{7}\Phi_{j,\,i}\cdot y_\Phi(i;\,t)$$

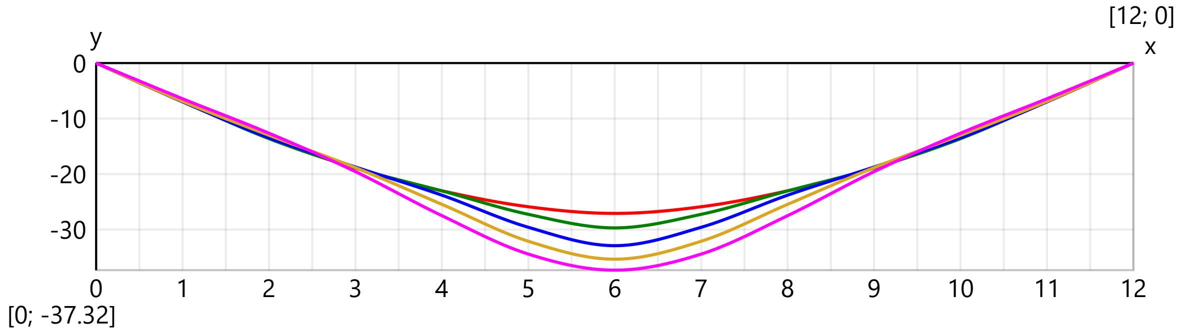
Comparison of time history records of the midpoint displacements for SDOF and MDOF systems, [mm]



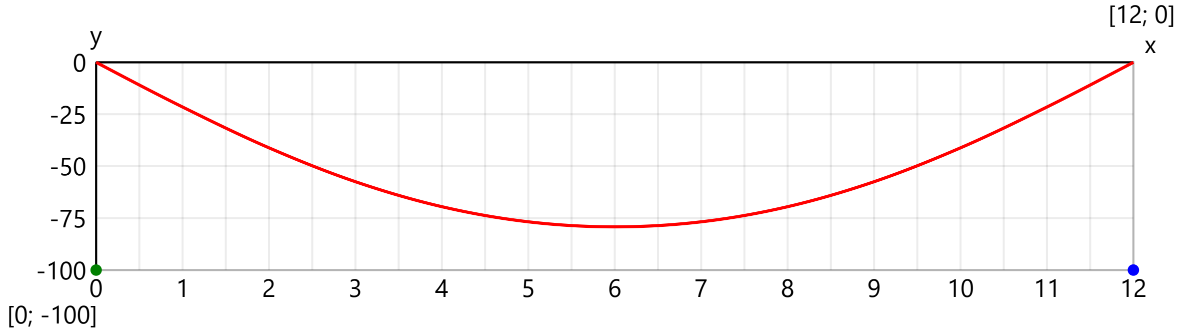
Time history records for the amplitudes of separate joints, [mm]



Beam deflections for the first five time steps at Δt = 17.7 ms



Animation of beam response (slowed down)



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